JOURNAL OF APPROXIMATION THEORY **58**, 12–14 (1989)

Generalization of an Inequality of Gronwall–Reid

DEAN S. CLARK

Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881, U.S.A.

Communicated by Oved Shisha

Received June 17, 1985; revised April 11, 1986

The inequality of Gronwall–Reid [1, 3] (with inspiration from Peano [2]) is a multi-purpose integral inequality most frequently used in differential equations. Here we give a new proof whose value lies in its easy generalization to the case where the Riemann differential "dt" is replaced by a positive measure. In particular, relation (3), below, shows that the classical Gronwall inequality is a simplest case of this more general type of inequality. We comment on the applications after stating our theorems.

THEOREM 1 (Peano–Gronwall–Reid). *Let f, α, and β be continuous real-valued functions defined on $[t_0, +\infty)$, $\beta \geqslant 0$. Suppose that for all $t \geqslant t_0$*

$$f(t) \leqslant \alpha(t) + \int_{t_0}^{t} \beta(s) f(s)\, ds. \tag{1a}$$

For $T > t_0$ let $S(t_0, T) = \{t$ where $f(t) \exp(-\int_{t_0}^{t} \beta(s)\, ds)$ is maximized in $[t_0, T]\}$. Then,

$$f(t) \leqslant \alpha^* \exp\left(\int_{t_0}^{t} \beta(s)\, ds\right), \qquad t_0 \leqslant t \leqslant T, \tag{1b}$$

where $\alpha^* = \min\{\alpha(\theta) \colon \theta \in S(t_0, T)\}$.

Proof. Multiply across (1a) by $\exp(-\int_{t_0}^{t} \beta(\sigma)\, d\sigma)$ and set $g(t) = f(t) \exp(-\int_{t_0}^{t} \beta(\sigma)\, d\sigma)$. We get

$$g(t) \leqslant \alpha(t) \exp\left(-\int_{t_0}^{t} \beta(\sigma)\, d\sigma\right) + \int_{t_0}^{t} g(s)\, d\left\{\exp\left(-\int_{s}^{t} \beta(\sigma)\, d\sigma\right)\right\}, \qquad t_0 \leqslant t. \tag{2}$$

To relate to our forthcoming generalizations we abbreviate (2) as

$$g(t) \leqslant \bar{\alpha}(t) + \varepsilon \int_{t_0}^{t} g(s)\, d\mu_\lambda, \qquad \varepsilon = 1. \tag{3}$$

0021-9045/89 $3.00

The multi-index λ in (3) is used to emphasize the dependence of the Stieltjes differentials du_λ on t and the function β. Observe that the μ_λ induce degenerate probability measures on $[t_0, +\infty)$.

For any $\theta \in S(t_0, T)$

$$g(t) \leqslant \bar{\alpha}(t) + g(\theta) \int_{t_0}^{t} d\mu_\lambda$$

is a valid inequality for all t in $[t_0, T]$. In particular, it holds at any θ^* which minimizes $\alpha(\cdot)$ on $S(t_0, T)$. Setting $t = \theta^*$ and integrating,

$$g(\theta^*) \leqslant \bar{\alpha}(\theta^*) + g(\theta^*)\left(1 - \exp\left(-\int_{t_0}^{\theta^*} \beta(\sigma)\, d\sigma\right)\right).$$

After cancellations we get $g(t) \leqslant g(\theta^*) \leqslant \alpha(\theta^*)$, $t_0 \leqslant t \leqslant T$, which is (1b).

THEOREM 2. *Let α, f, and β_j, $j = 1, ..., n$, be nonnegative functions defined on $[t_0, +\infty)$ which are bounded on finite intervals, f and the β_j measurable with respect to P an arbitrary probability measure on $[t_0, +\infty)$. Suppose that*

$$f(t) \leqslant \alpha(t) + \sum_{j=1}^{n} \int_{t_0}^{t} \beta_j(s) f^{q_j}(s)\, dP + \varepsilon \int_{t_0}^{t} f(s)\, dP, \qquad t_0 \leqslant t, \tag{4}$$

where $0 \leqslant q_j \leqslant \frac{1}{2}$ and $0 < \varepsilon < 1$. Then, for any $T > t_0$, there is $\theta(t_0, T) \in [t_0, T]$ such that

$$f(t) \leqslant \kappa_1 \overline{\lim_{s \to \theta}}\, \alpha(s) + \kappa_2 \sum_{j=1}^{n} \int_{t_0}^{\theta +} \beta_j^{p_j}(s)\, dP, \qquad t_0 \leqslant t \leqslant T, \tag{5}$$

where $\kappa_1 = 2/(1-\varepsilon)$, $\kappa_2 = 4n/(1-\varepsilon)^2$, and $p_j = 1/(1-q_j)$, $j = 1, ..., n$.

THEOREM 3. *Let μ_{λ_j}, $j = 1, ..., n$, denote finite measures on $[t_0, +\infty)$ and let f be a nonnegative function integrable on finite intervals with respect to every μ_{λ_j}. Suppose that*

$$f(t) \leqslant \alpha + \beta \sum_{j=1}^{m} \int_{t_0}^{t} f^{q_j}(s)\, d\mu_{\lambda_j} + \varepsilon \sum_{j=m+1}^{n} \int_{t_0}^{t} f(s)\, d\mu_{\lambda_j}, \qquad t_0 \leqslant t, \tag{6}$$

where $\alpha, \beta \geqslant 0$ are constants and $0 \leqslant q_j \leqslant \frac{1}{2}$. If $\mu_{\lambda_j}[t_0, +\infty) \leqslant M < +\infty$ for all multi-index values λ_j, and $0 < \varepsilon < 1/nM$, then

$$f(t) \leqslant \frac{\alpha + \beta c m M}{1 - nM(\varepsilon + \beta c^{-1})}, \qquad t_0 \leqslant t, \tag{7}$$

where $c \geqslant 1$ is any number greater than $\beta nM/(1 - \varepsilon nM)$.

Proof of Theorem 2. The proof of Theorem 3 is analogous. Use the well known inequality $xy \leqslant cx^p + (1/c)\, y^q$ for $x, y \geqslant 0$, $c \geqslant 1$, $q \geqslant 2$, and $1/p + 1/q = 1$ in (4) on $\beta_j(s) f^{q_j}(s)$ with $q = 1/q_j$, $p = 1/(1-q_j)$. As a result,

$$f(t) \leqslant \alpha(t) + c \sum_{j=1}^{n} \int_{t_0}^{t} \beta_j^{p_j}(s)\, dP + \left(\frac{n}{c} + \varepsilon\right) \int_{t_0}^{t} f(s)\, dP. \tag{8}$$

Take $\theta \in [t_0, T]$ so that $\overline{\lim}_{s \to \theta} f(s) = \sup_{t_0 \leqslant s \leqslant T} f(s)$. Applying the lim sup across (8) as $t \to \theta$ and consolidating terms gives (5), with $\kappa_1(c) = 1/(1 - \varepsilon - nc^{-1})$ and $\kappa_2(c) = c\kappa_1(c)$, where c is any number greater than $n/(1-\varepsilon)$. The θ^+ denotes the limit from the right. Minimizing $\kappa_2(c)$ with respect to c gives the final form of (5). The same minimization can be performed on the r.h.s. of (7).

Remark on the Applications. We have used Theorem 3 effectively in the stability analysis of the system of difference equations $X_{n+1} = X_n + E_n F_n(X_n, \Phi_n)$ with perturbations Φ_n, where the E_n are diagonal matrices with positive terms along the diagonal. Such systems model general recursive adjustment procedures under uncertainty when $\{\Phi_n\}_{n=0}^{\infty}$ is a stochastic process (see [4]).

REFERENCES

1. T. H. GRONWALL, Note on the derivatives with respect to a parameter of the solutions of a system of differential equations, *Ann. of Math.* **20**, No. 2 (1919), 292–296.
2. G. PEANO, Sull' integrabilita delle equazioni differenziali del primo ordine, *Atti R. Accad. Sci. Torino* **21** (1885–1886), 677–685.
3. W. T. REID, Properties of solutions of an infinite system of ordinary linear differential equations of the first order with auxiliary boundary conditions (Dissertation, University of Texas, 1929), *Trans. Amer. Màth. Soc.* **32** (1930), 284–318.
4. D. S. CLARK, Stability in a system of difference equations, *Quart. Appl. Math.* **45** (1987), 441–453.